THE SQUARE SIEVE AND THE LARGE SIEVE WITH SQUARE MODULI

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Abstract: We give a short alternative proof using Heath-Brown's square sieve of a bound of the author for the large sieve with square moduli.

1. Main result

The large sieve with square moduli and, more generally, power moduli was studied by L. Zhao and the author in a number of papers, both independently and in joint work. The best result for square moduli obtained so far is [2, Theorem 1] which asserts the following.

Theorem 1. *Let $\varepsilon > 0$. Then for any $M \in \mathbb{Z}$, $N \in \mathbb{N}$, $Q \geq 1$ and sequence of complex numbers $(a_n)_{n\in\mathbb{Z}}$, we have*

$$\sum_{q\leq Q} \sum_{\substack{a=1\\(a,q)=1}}^{q^2} \left|S\left(\frac{a}{q^2}\right)\right|^2 = O\left((NQ)^{\varepsilon}\left(Q^3+N+\min\left\{N\sqrt{Q},\sqrt{N}Q^2\right\}\right)Z\right), \tag{1}$$

where

$$S(\alpha) := \sum_{n=M+1}^{M+N} a_n e(n\alpha) \quad \textit{and} \quad Z := \sum_{n=M+1}^{M+N} |a_n|^2. \tag{2}$$

In [5], Zhao proved (1) with $N\sqrt{Q}+\sqrt{N}Q^2$ in place of the minimum of these terms using Fourier analysis. By combinatorial considerations, the author of the present paper then showed in [1] that the term $N\sqrt{Q}$ in the above sum can be omitted. Finally, combining their methods and making further refinements, both authors together succeeded in proving (1). We note that the two terms in the minimum coincide and give a contribution of $Q^{7/2}$ if $Q^3 = N$. It was conjectured by Zhao that (1) should hold without the minimum term.

The purpose of this paper is to give a short alternative proof of the bound

$$\sum_{q\leq Q} \sum_{\substack{a=1\\(a,q)=1}}^{q^2} \left|S\left(\frac{a}{q^2}\right)\right|^2 = O\left((NQ)^{\varepsilon}\left(Q^3+N+\sqrt{N}Q^2\right)Z\right), \tag{3}$$

previously obtained in [1], using Heath-Brown's square sieve. We note that the left-hand side of (3) is bounded by

$$\sum_{q\leq Q} \sum_{\substack{a=1\\(a,q)=1}}^{q^2} \left|S\left(\frac{a}{q^2}\right)\right|^2 \ll \sum_{q\leq Q^2} \sum_{\substack{a=1\\(a,q)=1}}^{q} \left|S\left(\frac{a}{q}\right)\right|^2 \ll \left(Q^4+N\right)Z \tag{4}$$

using the classical large sieve inequality (see, for example, Zhao's thesis [5] for reference), and also by

$$\sum_{q\leq Q} \sum_{\substack{a=1\\(a,q)=1}}^{q^2} \left|S\left(\frac{a}{q^2}\right)\right|^2 \ll Q(N+Q^2)Z, \tag{5}$$

which is obtained by summing over q the large sieve bound

$$\sum_{a=1}^{q} \left|S\left(\frac{a}{q}\right)\right|^2 \ll (q+N)\,Z$$

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for individual moduli (which can be seen by just expanding the square and re-arranging summations). Since Zhao's conjecture follows from (4) if $Q \le N^{1/4+\varepsilon}$ and from (5) if $Q \ge N^{1/2-\varepsilon}$, we shall assume that

$$N^{1/4+\varepsilon} < Q < N^{1/2-\varepsilon} \tag{6}$$

throughout this paper. We note that (3) is the same as (1) if $Q \le N^{1/3}$.

Notation and conventions:

(i) We keep the notations in Theorem 1, in particular those in (2), throughout this paper.
(ii) $n = \square$ means that n is a square.
(iii) $d(n) := \sum_{m|n} 1$ denotes the divisor function.
(iv) ε is an arbitrarily small positive number which can change from line to line.

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2. Preliminaries

Similarly as in [1], our starting point will be the following Lemma which is a direct consequence of [4, Theorem 2.1].

Lemma 1. *Assume that $Q \ge 1$, $N \ge 1$ and $0 < \Delta \le 1$. Then*

$$\sum_{Q<q\le 2Q} \sum_{\substack{a=1\\(a,q)=1}}^{q^2} \left| S\left(\frac{a}{q^2}\right)\right|^2 \ll \left(N+\Delta^{-1}\right) Z \cdot \max_{\alpha\in\mathbb{R}} P\left(\alpha,\Delta\right), \tag{7}$$

where

$$P(\alpha,\Delta) := \sum_{\substack{Q<q\le 2Q}} \sum_{\substack{a=1\\(a,q)=1\\|a/q^2-\alpha|\le\Delta}}^{q^2} 1. \tag{8}$$

To detect squares, we shall employ Heath-Brown's square sieve (see [3, Theorem 1]).

Lemma 2. *Let $\mathcal{P}$ be a set of P primes. Suppose that $w : \mathbb{Z} \to \mathbb{R}$ is a function satisfying $w(n) \ge 0$ for all $n \in \mathbb{N}$ and $w(n) = 0$ if $n = 0$ or $|n| \ge e^P$. Then*

$$\sum_{n=1}^{\infty} w\left(n^2\right) \ll P^{-1} \sum_{n\in\mathbb{Z}} w(n) + P^{-2} \sum_{\substack{p_1,p_2\in\mathcal{P}\\p_1\ne p_2}} \left| \sum_{n\in\mathbb{Z}} w(n) \left(\frac{n}{p_1p_2}\right)\right|,$$

where $\left(\frac{n}{p_1p_2}\right)$ is the Jacobi symbol.

3. Division into major and minor arcs

Dividing the q-range in (1) into dyadic intervals, it suffices to prove that

$$\sum_{Q<q\le 2Q} \sum_{\substack{a=1\\(a,q)=1}}^{q^2} \left| S\left(\frac{a}{q^2}\right)\right|^2 = O\left((NQ)^{\varepsilon}\left(Q^3+N+\sqrt{N}Q^2\right)Z\right). \tag{9}$$

Our task is now to estimate the term $P(\alpha,\Delta)$ in Lemma 1 for any given $\alpha \in \mathbb{R}$. We shall choose

$$\Delta := \frac{1}{N}. \tag{10}$$

First, we first consider α in a set of major arcs, defined by

$$\mathfrak{M} := \bigcup_{v \le 1/(500Q^2\Delta)} \bigcup_{\substack{u=1\\(u,v)=1}}^{v} \left[\frac{u}{v} - \frac{1}{10Q^2v}, \frac{u}{v} + \frac{1}{10Q^2v}\right].$$

If $\alpha \in \mathfrak{M}$, then there exist $u, v \in \mathbb{Z}$ such that $1 \le v \le 1/(500Q^2\Delta)$, $(u,v) = 1$ and $|u/v - \alpha| \le 1/(10Q^2v)$ and hence

$$P(\alpha, \Delta) \le P\left(\frac{u}{v}, \frac{1}{5Q^2v}\right) = \sum_{Q<q\le 2Q} \sum_{\substack{a=1\\(a,q)=1\\|a/q^2-u/v|\le 1/(5Q^2v)}}^{q^2} 1 \le \sum_{Q<q\le 2Q} \sum_{\substack{a=1\\(a,q)=1\\|av-uq^2|<1}}^{q^2} 1 = \sum_{Q<q\le 2Q} \sum_{\substack{a=1\\(a,q)=1\\a/q^2=u/v}}^{q^2} 1 = 0 \quad (11)$$

since $(a,q^2) = 1 = (u,v)$ and $v \le 1/(500Q^2\Delta) < Q^2 < q^2$ by (6) if N is large enough. Hence, the major arcs don't contribute.

In the remainder, we consider the case when α is in the set of minor arcs, defined by

$$\mathfrak{m} := \mathbb{R} \setminus \mathfrak{M}.$$

4. Application of the square sieve

By Dirichlet's approximation theorem, there exist integers b and r such that

$$1 \le r \le 500Q^2, \quad (b,r) = 1 \quad \text{and} \quad \left|\frac{b}{r} - \alpha\right| \le \frac{1}{500Q^2r}. \quad (12)$$

If $r \le 1/(500Q^2\Delta)$, then it follows that $\alpha \in \mathcal{M}$. Thus, (12) can be replaced by

$$1/(500Q^2\Delta) < r \le 500Q^2, \quad (b,r) = 1, \quad \text{and} \quad \left|\frac{b}{r} - \alpha\right| \le \Delta. \quad (13)$$

It follows that

$$P(\alpha, \Delta) \le P\left(\frac{b}{r}, 2\Delta\right). \quad (14)$$

Let Φ_1 and Φ_2 be infinitely differentiable compactly supported functions from $\mathbb{R}$ to $\mathbb{R}^+$, supported in the intervals $[1/2, 5]$ and $[-10, 10]$ and bounded below by 1 on the intervals $[1,4]$ and $[-4,4]$, respectively. Then

$$P\left(\frac{b}{r}, 2\Delta\right) \ll \sum_{q\in\mathbb{Z}} \Phi_1\left(\frac{q^2}{Q^2}\right) \cdot \sum_{a\in\mathbb{Z}} \Phi_2\left(\frac{a - q^2b/r}{Q^2\Delta}\right). \quad (15)$$

Let

$$R > (QN)^{\varepsilon} \quad (16)$$

be a parameter, to be fixed later, and

$$\mathcal{P} := \{p \in \mathbb{P} \ : \ R < p \le 2R \text{ and } p \nmid r\}, \quad (17)$$

where $\mathbb{P}$ is the set of all primes. In the notation of Lemma 2, we have

$$P := \sharp\mathcal{P} = \pi(2R) - \pi(R) - \omega(r) \sim \frac{R}{\log R}. \quad (18)$$

Now applying the square sieve, Lemma 2, to the right-hand side of (15), we get

$$\begin{aligned} P\left(\frac{b}{r}, 2\Delta\right) \ll & \frac{1}{P} \cdot \sum_{n\in\mathbb{Z}} \Phi_1\left(\frac{n}{Q^2}\right) \cdot \sum_{a\in\mathbb{Z}} \Phi_2\left(\frac{a - nb/r}{Q^2\Delta}\right) + \\ & \frac{1}{P^2} \cdot \sum_{\substack{p_1,p_2\in\mathcal{P}\\p_1\ne p_2}} \left|\sum_{n\in\mathbb{Z}} \Phi_1\left(\frac{n}{Q^2}\right) \cdot \left(\frac{n}{p_1p_2}\right) \cdot \sum_{a\in\mathbb{Z}} \Phi_2\left(\frac{a - bn/r}{Q^2\Delta}\right)\right|. \end{aligned} \quad (19)$$

The first double sum over n and a on the right-hand side of (19) can be estimated by

$$\sum_{n\in\mathbb{Z}} \Phi_1\left(\frac{n}{Q^2}\right)\cdot \sum_{a\in\mathbb{Z}} \Phi_2\left(\frac{a-nb/r}{Q^2\Delta}\right) \le \sum_{\substack{Q^2/2\le n\le 5Q^2}} \sum_{\substack{a\in\mathbb{Z}\\ |a/n-b/r|\le 20\Delta}} 1$$

$$= \sum_{Q^2/2\le n\le 5Q^2} \sum_{\substack{a\in\mathbb{Z}\\ (a,n)\le 2500Q^4\Delta\\ |a/n-b/r|\le 20\Delta}} 1 + \sum_{Q^2/2\le n\le 5Q^2} \sum_{\substack{a\in\mathbb{Z}\\ (a,n)>2500Q^4\Delta\\ |a/n-b/r|\le 20\Delta}} 1$$

$$\le \sum_{d\le 2500Q^4\Delta} \sum_{Q^2/(2d)\le n_1\le 5Q^2/d} \sum_{\substack{a_1\in\mathbb{Z}\\ (a_1,n_1)=1\\ |a_1/n_1-b/r|\le 20\Delta}} 1 + \sum_{n_1\le 1/(500Q^2\Delta)} \sum_{\substack{a_1\in\mathbb{Z}\\ (a_1,n_1)=1\\ |a_1/n_1-b/r|\le 20\Delta}} \sum_{Q^2/(2n_1)\le d\le 5Q^2/n_1} 1.$$

Since $|a_1/n_1 - a_2/n_2| \ge d^2/(25Q^4)$ whenever $Q^2/(2d) \le n_1, n_2 \le 5Q^2/d$, $(a_1, n_1) = 1 = (a_2, n_2)$ and $a_1/n_1 \ne a_2/n_2$, it follows that

$$\sum_{d\le 2500Q^4\Delta} \sum_{Q^2/(2d)\le n_1\le 5Q^2/d} \sum_{\substack{a_1\in\mathbb{Z}\\ (a_1,n_1)=1\\ |a_1/n_1-b/r|\le 20\Delta}} 1 \ll \sum_{d\le 2500Q^4\Delta} \left(1+\frac{Q^4\Delta}{d^2}\right) \ll Q^4\Delta.$$

Further,

$$\sum_{n_1\le 1/(500Q^2\Delta)} \sum_{\substack{a_1\in\mathbb{Z}\\ (a_1,n_1)=1\\ |a_1/n_1-b/r|\le 20\Delta}} \sum_{Q^2/(2n_1)\le d\le 5Q^2/n_1} 1 = 0$$

since $|a_1/n_1 - b/r| \le 20\Delta$ implies $|a_1/n_1 - \alpha| \le 21\Delta$ by (13), and hence, $\alpha \in \mathfrak{M}$, which is not the case. Altogether, we thus obtain

$$\sum_{n\in\mathbb{Z}} \Phi_1\left(\frac{n}{Q^2}\right)\cdot \sum_{a\in\mathbb{Z}} \Phi_2\left(\frac{a-nb/r}{Q^2\Delta}\right) \ll Q^4\Delta. \tag{20}$$

5. Application of Poisson summation

Now we estimate the second double sum over n and a on the right-hand side of (19). We first split the summation over n into residue classes modulo p_1p_2r, getting

$$\begin{aligned} &\sum_{n\in\mathbb{Z}} \Phi_1\left(\frac{n}{Q^2}\right)\cdot \left(\frac{n}{p_1p_2}\right)\cdot \sum_{a\in\mathbb{Z}} \Phi_2\left(\frac{a-bn/r}{Q^2\Delta}\right)\\ =&\sum_{f=1}^{p_1p_2r} \left(\frac{f}{p_1p_2}\right)\cdot \sum_{(x_1,x_2)\in\mathbb{Z}^2} \Phi_1\left(\alpha_1(f+mx_1)\right)\cdot \Phi_2\left(\beta_2x_2+\alpha_2(f+mx_1)\right), \end{aligned} \tag{21}$$

where

$$\alpha_1 := \frac{1}{Q^2}, \quad \beta_2 := \frac{1}{Q^2\Delta}, \quad \alpha_2 := -\frac{b}{rQ^2\Delta}, \quad m := p_1p_2r. \tag{22}$$

Now we apply the two-dimensional Poisson summation

$$\sum_{(x_1,x_2)\in\mathbb{Z}} \Phi(x_1,x_2) = \sum_{(t_1,t_2)\in\mathbb{Z}} \hat{\Phi}(t_1,t_2) \tag{23}$$

to the function

$$\Phi(x_1,x_2) := \Phi_1\left(\alpha_1(f+mx_1)\right)\cdot \Phi_2\left(\beta_2x_2+\alpha_2(f+mx_1)\right) \tag{24}$$

whose Fourier transform is

$$\hat{\Phi}(t_1,t_2) = \frac{1}{\beta_2\alpha_1 m}\cdot e\left(\frac{lt_1}{m}\right)\cdot \hat{\Phi}_1\left(\frac{t_1}{\alpha_1 m} - \frac{t_2\alpha_2}{\alpha_1\beta_2}\right)\cdot \hat{\Phi}_2\left(\frac{t_2}{\beta_2}\right). \tag{25}$$

Combining (21), (22), (23), (24) and (25), we deduce that

$$\begin{aligned}&\sum_{n\in\mathbb{Z}}\Phi_1\left(\frac{n}{Q^2}\right)\cdot\left(\frac{n}{p_1p_2}\right)\cdot\sum_{a\in\mathbb{Z}}\Phi_2\left(\frac{a-bn/r}{Q^2\Delta}\right)\\ =&\frac{Q^4\Delta}{p_1p_2r}\cdot\sum_{c\in\mathbb{Z}}\hat{\Phi}_2\left(cQ^2\Delta\right)\cdot\sum_{s\in\mathbb{Z}}\hat{\Phi}_1\left(\frac{sQ^2}{p_1p_2r}\right)\sum_{f=1}^{p_1p_2r}\left(\frac{f}{p_1p_2}\right)\cdot e\left(\frac{(s-cbp_1p_2)f}{p_1p_2r}\right).\end{aligned}\tag{26}$$

Since $(r,p_1p_2)=1$, we have

$$\begin{aligned}\sum_{f=1}^{p_1p_2r}\left(\frac{f}{p_1p_2}\right)\cdot e\left(\frac{(s-cbp_1p_2)f}{p_1p_2r}\right)=&\sum_{g=1}^{p_1p_2}\sum_{h=1}^{r}\left(\frac{gr}{p_1p_2}\right)\cdot e\left(\frac{(s-cbp_1p_2)(gr+hp_1p_2)}{p_1p_2r}\right)\\ =&\sum_{g=1}^{p_1p_2}\left(\frac{gr}{p_1p_2}\right)\cdot e\left(\frac{sg}{p_1p_2}\right)\cdot\sum_{h=1}^{r}e\left(\frac{(s-cbp_1p_2)h}{r}\right)\\ =&r\tau_{p_1p_2}\cdot\left(\frac{rs}{p_1p_2}\right)\cdot\begin{cases}1 & \text{if } cp_1p_2\equiv s\bar{b} \bmod r\\ 0 & \text{otherwise,}\end{cases}\end{aligned}\tag{27}$$

where $\tau_{p_1p_2}$ is the Gauss sum for the Jacobi symbol modulo p_1p_2. Combining (26) and (27), and using the triangle inequality and the well-known equation $|\tau_{p_1p_2}|=\sqrt{p_1p_2}$, we get

$$\begin{aligned}&\left|\sum_{n\in\mathbb{Z}}\Phi_1\left(\frac{n}{Q^2}\right)\cdot\left(\frac{n}{p_1p_2}\right)\cdot\sum_{a\in\mathbb{Z}}\Phi_2\left(\frac{a-bn/r}{Q^2\Delta}\right)\right|\\ \le&\frac{Q^4\Delta}{\sqrt{p_1p_2}}\cdot\sum_{\substack{c\in\mathbb{Z}\\ cp_1p_2\equiv s\bar{b} \bmod r}}\sum_{s\in\mathbb{Z}}\left|\hat{\Phi}_1\left(\frac{sQ^2}{p_1p_2r}\right)\cdot\hat{\Phi}_2\left(cQ^2\Delta\right)\right|.\end{aligned}\tag{28}$$

6. Completion of the proof

Let $T=(QRN)^{\varepsilon}$. Summing the right-hand side of (28) over p_1,p_2, and using the rapid decays of $\hat{\Phi}_1$ and $\hat{\Phi}_2$, we get

$$\begin{aligned}&\sum_{\substack{p_1,p_2\in\mathcal{P}\\ p_1\neq p_2}}\frac{Q^4\Delta}{\sqrt{p_1p_2}}\cdot\sum_{\substack{c\in\mathbb{Z}\\ cp_1p_2\equiv s\bar{b} \bmod r}}\sum_{s\in\mathbb{Z}}\left|\hat{\Phi}_1\left(\frac{sQ^2}{p_1p_2r}\right)\cdot\hat{\Phi}_2\left(cQ^2\Delta\right)\right|\\ \ll&1+\frac{Q^4\Delta}{R}\cdot\sum_{R^2<m\le(2R)^2}\sum_{\substack{|c|\le T/(Q^2\Delta)\\ cm\equiv s\bar{b} \bmod r}}\sum_{|s|\le TR^2r/Q^2}1\\ \ll&1+\frac{Q^4\Delta}{R}\cdot\left(R^2\cdot\left(1+\frac{TR^2}{Q^2}\right)+\sum_{|s|\le TR^2r/Q^2}\sum_{\substack{R^2<m\le(2R)^2\\ cm\equiv s\bar{b} \bmod r}}\sum_{1\le|c|\le T/(Q^2\Delta)}1\right)\\ \ll&1+Q^4\Delta R+TQ^2\Delta R^3+\frac{Q^4\Delta}{R}\cdot\sum_{|s|\le TR^2r/Q^2}\sum_{\substack{1\le t\le 4R^2T/(Q^2\Delta)\\ t\equiv s\bar{b} \bmod r}}d(t)\\ \ll&1+Q^4\Delta R+TQ^2\Delta R^3+\frac{TQ^4\Delta}{R}\cdot\left(1+\frac{TR^2r}{Q^2}\right)\cdot\left(1+\frac{R^2T}{rQ^2\Delta}\right)\\ \ll&1+T^3\left(Q^4\Delta R+Q^2\Delta R^3+\frac{Q^4\Delta}{R}+Q^2\Delta Rr+\frac{Q^2R}{r}+R^3\right)\\ \ll&1+T^3\left(Q^4\Delta R+Q^2\Delta R^3+R^3\right),\end{aligned}\tag{29}$$

where for the last line, we use the bounds for r in (13) and (16). Combining (18), (19), (20), (28) and (29), we get

$$P\left(\frac{b}{r}, 2\Delta\right) \ll (QRN)^{\varepsilon} \cdot \left(\frac{Q^4\Delta}{R} + Q^2\Delta R + R\right).$$

Now we choose

$$R := Q^2\sqrt{\Delta}$$

which is consistent with (16) by (6) and (10) and gives

$$P\left(\frac{b}{r}, 2\Delta\right) \ll (QN)^{\varepsilon} \cdot \left(Q^2\Delta^{1/2} + Q^4\Delta^{3/2}\right).$$

Plugging this into (7) and using (6) and (10) proves (3). □

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